

THE QUANTUM GALILEI GROUP

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ABSTRACT. The quantum Galilei group G_{\varkappa} is defined. The bicrossproduct structure of G_{\varkappa} and the corresponding Lie algebra is revealed. The projective representations for twodimensional quantum Galilei group are constructed.

I. INTRODUCTION

Some attention has been recently paid to the so-called deformed Poincaré algebra [1] and group [2]. They are interesting because they provide relatively mild deformation of classical space-time symmetries depending on dimensionful deformation parameter. It is therefore interesting to study in more detail some properties of the resulting structure. In particular, in Ref. [3] some preliminary remarks have been made concerning the non-relativistic limit of deformed symmetry.

In the present paper we continue this analysis. We construct the \varkappa -deformed Galilei group and reveal the bicrossproduct structure of both the algebra and the group (cf. Ref. [4]).

We address also to the question of the quantum counterpart of projective representations. They are defined in analogy with classical case. For twodimensional deformed Galilei group the projective representations are explicitly constructed.

II. THE DEFORMED GALILEAN ALGEBRA

The deformed Galilean algebra \mathfrak{g}_{\varkappa} can be obtained by contraction procedure from k -Poincaré algebra in the way described in Ref. [3]. We make the rescaling $P_0 \rightarrow P_0 \cdot c$, $L_i \rightarrow L_i \cdot c^{-1}$ and let $c \rightarrow \infty$ keeping $kc \equiv \varkappa = \text{const.}$ The resulting

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structure reads :

$$\begin{aligned}
[M_i, P_j] &= i\varepsilon_{ijk}P_k, & [M_i, P_0] &= 0, \\
[M_i, M_j] &= i\varepsilon_{ijk}M_k, & [P_\mu, P_\nu] &= 0, \\
[M_i, L_j] &= i\varepsilon_{ijk}L_k, \\
[L_i, P_0] &= iP_i, & [L_i, P_k] &= 0, \\
[L_i, L_j] &= \frac{1}{4\kappa^2}\varepsilon_{ijk}P_k \left(\vec{P} \cdot \vec{M} \right), \\
\Delta M_i &= M_i \otimes I + I \otimes M_i, & \Delta P_0 &= P_0 \otimes I + I \otimes P_0, \\
\Delta P_i &= P_i \otimes e^{-\frac{P_0}{2\kappa}} + e^{\frac{P_0}{2\kappa}} \otimes P_i, \\
\Delta L_i &= L_i \otimes e^{-\frac{P_0}{2\kappa}} + e^{\frac{P_0}{2\kappa}} \otimes L_i - \frac{1}{2\kappa}\varepsilon_{ikl} \left(e^{\frac{P_0}{2\kappa}} M_k \otimes P_l + P_k \otimes e^{-\frac{P_0}{2\kappa}} M_l \right), \\
S(P_\mu) &= -P_\mu, & S(M_i) &= -M_i, \\
S(L_i) &= -L_i - \frac{3i}{2\kappa}P_i.
\end{aligned} \tag{1}$$

Let us note that this algebra is obtained by contraction accompanied with strong deformation limit $k \rightarrow 0$. It seems that there exists no nonrelativistic limit ($c \rightarrow \infty$) with k kept fixed [5].

The Casimir operators can be also obtained by contraction. They read :

$$\begin{aligned}
C_1 &= \vec{P}^2, \\
C_2 &= \frac{\vec{P}^2}{4\kappa^2} \left(\vec{P} \cdot \vec{M} \right)^2 + \left(\vec{P} \times \vec{L} \right)^2.
\end{aligned} \tag{2}$$

Obviously, in the limit $\kappa \rightarrow \infty$ we recover standard Galilean structure. As in the case of k -deformed Poincaré algebra [4] one can show that our algebra has a bicrossproduct structure. To this end we define new generators :

$$\begin{aligned}
\widetilde{M}_i &= M_i, & \widetilde{P}_0 &= P_0, \\
\widetilde{P}_i &= P_i e^{-\frac{P_0}{2\kappa}}, & \widetilde{L}_i &= L_i e^{-\frac{P_0}{2\kappa}} - \frac{\varepsilon_{ijk}}{2\kappa} M_j P_k e^{-\frac{P_0}{2\kappa}}.
\end{aligned} \tag{3}$$

Then we obtain

$$\mathfrak{g}_\kappa = T \bowtie U \left(\widetilde{M}, \widetilde{L} \right), \tag{4}$$

where $U \left(\widetilde{M}, \widetilde{L} \right)$ is universal covering of Lie algebra $\mathfrak{e}(3)$

$$\begin{aligned}
[\widetilde{M}_i, \widetilde{M}_j] &= i\varepsilon_{ijk}\widetilde{M}_k, & [\widetilde{M}_i, \widetilde{L}_j] &= i\varepsilon_{ijk}\widetilde{L}_k, & [\widetilde{L}_i, \widetilde{L}_j] &= 0, \\
\Delta \widetilde{M}_i &= \widetilde{M}_i \otimes I + I \otimes \widetilde{M}_i, & \Delta \widetilde{L}_i &= \widetilde{L}_i \otimes I + I \otimes \widetilde{L}_i
\end{aligned} \tag{5}$$

while T is defined by

$$\begin{aligned}
[\widetilde{P}_\mu, \widetilde{P}_\nu] &= 0, \\
\Delta \widetilde{P}_0 &= \widetilde{P}_0 \otimes I + I \otimes \widetilde{P}_0, & \Delta \widetilde{P}_i &= \widetilde{P}_i \otimes e^{-\frac{\widetilde{P}_0}{\kappa}} + I \otimes \widetilde{P}_i.
\end{aligned} \tag{6}$$

Equation (4) is readily verified once we define

$$\begin{aligned}
\widetilde{M}_i \triangleright \widetilde{P}_0 &= 0, & \widetilde{M}_i \triangleright \widetilde{P}_j &= i\varepsilon_{ijk}\widetilde{P}_k, & \widetilde{L}_i \triangleright \widetilde{P}_0 &= i\widetilde{P}_i, \\
\widetilde{L}_i \triangleright \widetilde{P}_j &= \frac{i}{2\kappa}\delta_{ij}\widetilde{P}^2 - \frac{i}{\kappa}\widetilde{P}_i\widetilde{P}_j, \\
\delta\left(\widetilde{M}_i\right) &= \widetilde{M}_i \otimes I, & \delta\left(\widetilde{L}_i\right) &= \widetilde{L}_i \otimes e^{-\frac{\widetilde{P}_0}{\kappa}} - \frac{i}{\kappa}\varepsilon_{ijk}\widetilde{M}_j \otimes \widetilde{P}_k.
\end{aligned} \tag{7}$$

Let us now define the cocommutator

$$\sigma = (\Delta - \tau \circ \Delta) \left(\text{mod } \frac{1}{\kappa} \right); \tag{8}$$

we obtain

$$\begin{aligned}
\sigma(P_0) &= 0, & \sigma(M_i) &= 0, \\
\sigma(P_i) &= -\frac{1}{\kappa}P_i \wedge P_0, \\
\sigma(L_i) &= -\frac{1}{\kappa}(L_i \wedge P_0 + \varepsilon_{ikl}M_k \wedge P_l).
\end{aligned} \tag{9}$$

However, σ is not a coboundary – the classical r -matrix does not exist. This can be verified by direct calculations.

III. THE DEFORMED GALILEI GROUP

The deformed Galilei group G_κ can be obtained either via contraction procedure or by quantizing the Poisson structure on classical group implied by equations (9). First we shall consider contraction procedure.

The relations defining classical Lorentz group, $g_{\mu\nu}\Lambda_\alpha^\mu\Lambda_\beta^\nu = g_{\alpha\beta}$, can be solved explicitly in terms of rotation matrix R_j^i and velocity v^i .

$$\begin{aligned}
\Lambda_0^0 &= \left(1 + \frac{\vec{v}^2}{c^2}\right)^{\frac{1}{2}}, \\
\Lambda_0^i &= \frac{v^i}{c}, \\
\Lambda_i^0 &= \frac{v^k R_i^k}{c}, \\
\Lambda_i^k &= \left(\delta_l^k + \left(\left(1 + \frac{\vec{v}^2}{c^2}\right)^{\frac{1}{2}} - 1\right) \frac{v^k v^l}{\vec{v}^2}\right) R_i^l.
\end{aligned} \tag{10}$$

If we further redefine the translation sector, $a^i \rightarrow a^i$, $a^0 \rightarrow c\tau$, and let $c \rightarrow \infty$, $kc \equiv \kappa = \text{const}$ in the commutation rules and coproduct defining the k -Poincaré

group [2] we arrive at the following structure

$$\begin{aligned}
[R_j^i, R_l^k] &= 0, & [R_j^i, v^k] &= 0, & [v^i, v^k] &= 0, \\
[a^i, a^j] &= 0, & [\tau, a^i] &= \frac{i}{\varkappa} a^i, \\
[\tau, v^i] &= \frac{i}{\varkappa} v^i, & [v^i, a^j] &= -\frac{i}{\varkappa} \left(v^i v^j - \frac{1}{2} \bar{v}^2 \delta_{ij} \right), \\
[R_j^i, \tau] &= 0, & [R_j^i, a^k] &= -\frac{i}{\varkappa} (v^i R_j^k - \delta_{ik} v^m R_j^m), \\
\Delta R_j^i &= R_k^i \otimes R_j^k, \\
\Delta v^i &= R_j^i \otimes v^j + v^i \otimes I, \\
\Delta \tau &= \tau \otimes I + I \otimes \tau, \\
\Delta a^i &= R_j^i \otimes a^j + v^i \otimes \tau + a^i \otimes I.
\end{aligned} \tag{11}$$

The same structure is obtained by quantizing the Poisson structure on classical Galilei group implied by the cocommutator σ (equation (9)). We define the Poisson bracket by

$$\langle \{f, g\}, X \rangle = -i \langle f \otimes g, \sigma(X) \rangle, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ defines the classical duality between Lie group and algebra which, in our case, reads

$$\begin{aligned}
\langle \tau, P_0 \rangle &= i, & \langle v^i, L_k \rangle &= -i \delta_k^i, \\
\langle a^i, P_k \rangle &= -i \delta_k^i, & \langle R_j^i, M_k \rangle &= -i \varepsilon_{ijk},
\end{aligned} \tag{13}$$

the remaining brackets being zero. A straightforward calculations (see Appendix) allow us to recover the commutation rules (11).

Again, one can equip G_\varkappa with bicrossproduct structure. Namely

$$G_\varkappa = T^* \bowtie C(E(3)), \tag{14}$$

where $C(E(3))$ is the algebra of functions on classical group $E(3)$, generated by R_j^i and v^i while T^* is defined by

$$\begin{aligned}
[\tau, a^i] &= \frac{i}{\varkappa} a^i, & [a^i, a^j] &= 0, \\
\Delta a^i &= a^i \otimes I + I \otimes a^i, & \Delta \tau &= \tau \otimes I + I \otimes \tau;
\end{aligned} \tag{15}$$

moreover,

$$\begin{aligned}
f \triangleleft g &= [f, g], & f &\in C(E(3)), \quad g \in T^*, \\
\beta(\tau) &= I \otimes \tau, \\
\beta(a^i) &= R_j^i \otimes a^j + v^i \otimes \tau.
\end{aligned} \tag{16}$$

It is likely that the recent proof of duality for k -Poincaré algebra and group [6], [7] can be adapted to our pair $(G_\varkappa, \mathfrak{g}_\varkappa)$.

As next step we define the Galilean space-time. It is algebra generated by four elements t, x^i , subject to the following commutation rules

$$[t, x^i] = \frac{i}{\varkappa} x^i, \quad [x^i, x^j] = 0. \tag{17}$$

The Galilei group G_\varkappa acts covariantly according to the rules

$$\begin{aligned}
t &\rightarrow I \otimes t + \tau \otimes I, \\
x^i &\rightarrow R_j^i \otimes x^j + v^i \otimes t + a^i \otimes I.
\end{aligned} \tag{18}$$

IV. PROJECTIVE REPRESENTATIONS

It is well known that the Galilei group posses one-parameter family of projective representations [8]. Moreover, they are exactly those representations which have physical meaning; the parameter labelling inequivalent projective representations is simply the mass of a particle. So the natural question arises whether the projective representations have some counterpart in the deformed case.

We have at our disposal no general theory of projective representations for quantum groups. However, we can try to follow closely the classical case and define an unitary projective representation of the quantum group \mathcal{A} as a map $\varrho : H \rightarrow H \otimes \mathcal{A}$ satisfying

$$(\varrho \otimes I) \circ \varrho(\psi) = (I \otimes \omega)((I \otimes \Delta) \circ \varrho(\psi)), \quad \psi \in H; \quad (19)$$

here H is the relevant Hilbert space of states and ω is an unitary element of $\mathcal{A} \otimes \mathcal{A}$. As in the classical case one easily derives the following consistency condition on ω (implied by (co-)associativity)

$$(\omega \otimes I)(\Delta \otimes I)\omega = (I \otimes \omega)(I \otimes \Delta)\omega. \quad (20)$$

Two projective representations $\varrho, \tilde{\varrho}$, are called equivalent if there exists an unitary element $a \in \mathcal{A}$ such that

$$\tilde{\varrho} = (I \otimes a)\varrho. \quad (21)$$

The corresponding multipliers are related by the formula

$$(a \otimes a)\omega = \tilde{\omega}\Delta(a). \quad (22)$$

Two such multipliers will be called equivalent. Obviously, a multiplier ω is trivial (the representation is equivalent to the vector one) if

$$(a \otimes a)\omega = \Delta(a). \quad (23)$$

So, the problem reduces to that of finding solutions of equation (20) which are not of the form (23). We shall solve this problem for twodimensional Galilei group as a toy model.

The dimensional reduction applied to G_{κ} gives rise to the following structure

$$\begin{aligned} [\tau, a] &= \frac{i}{\kappa}a, & [\tau, v] &= \frac{i}{\kappa}v, & [v, a] &= -\frac{i}{2\kappa}v^2, \\ \Delta v &= v \otimes I + I \otimes v, & \Delta \tau &= \tau \otimes I + I \otimes \tau, \\ \Delta a &= a \otimes I + I \otimes a + v \otimes \tau. \end{aligned} \quad (24)$$

For the the classical group the multiplier takes the form

$$\omega_0 = e^{i\varphi_0}, \quad \varphi_0 = -m \left(\frac{v^2}{2} \otimes \tau + v \otimes a \right) \quad (25)$$

Therefore we assume the same exponential form for our quantum multiplier

$$\omega = e^{i\varphi} \quad (26a)$$

and expand φ in inverse powers of \varkappa :

$$\varphi = \sum_{n=0}^{\infty} \frac{\varphi_n}{\varkappa^n} \quad (26b)$$

To provide nontriviality of ω we choose the first term in the expansion to be given by equation (25).

Inserting (26a), (26b) into equation (20) and comparing the coefficients in front of $\frac{1}{\varkappa}$ we arrive at the equation for φ_1 . The contribution to both sides are twofold :

- (i) the ones coming from φ_1 and
- (ii) those coming from the commutators of φ_0 due to Hausdorff formula.

Let us note that after taking commutators all elements can be viewed as classical ones – the contributions due to the noncommutativity are of higher orders in $\frac{1}{\varkappa}$. The resulting equation reads

$$\begin{aligned} \varphi_1 \otimes I - I \otimes \varphi_1 + (\Delta \otimes I)\varphi_1 - (I \otimes \Delta)\varphi_1 \\ = \frac{i\varkappa}{2}([I \otimes \varphi_0, (I \otimes \Delta)\varphi_0] - [\varphi_0 \otimes I, (\Delta \otimes I)\varphi_0]), \end{aligned} \quad (27)$$

where, after calculating the commutators on right-hand side the equation can be viewed as defined on classical Galilei group.

With some effort we can solve equation (27) (which is typical cohomological equation) for φ_1 . The particular solution reads

$$\varphi_1 = \left(-\frac{1}{4} \frac{mv^2}{\varkappa} \otimes I \right) \varphi_0. \quad (28)$$

Therefore we assume the following Ansatz for φ :

$$\varphi = (f(v^2) \otimes I) \varphi_0. \quad (29)$$

By inserting (29) and (26a) into equation (20) we find after some tedious calculations that this Ansatz is consistent and provides the following expression for ω :

$$\omega = e^{-i\left(\frac{2\varkappa}{v^2} \ln\left(1 + \frac{mv^2}{2\varkappa}\right) \otimes I\right) \left(\frac{v^2}{2} \otimes \tau + v \otimes a\right)} \quad (30)$$

or, equivalently

$$\omega = e^{-i\varkappa \ln\left(1 + \frac{mv^2}{2\varkappa}\right) \otimes \tau} e^{\frac{-imv}{1 + \frac{mv^2}{2\varkappa}} \otimes a}. \quad (31)$$

In order to obtain the relevant representation we can argue as follows. In the classical case, given the multiplier $\omega(g, g')$ one can construct the relevant representation by considering the linear space of functions defined over the group and defining the group action by the formula : $f(g) \rightarrow \omega(g, g')f(gg')$. Equation (20) allow us to conclude that this is projective representation, $\omega(g, g')$ being the relevant multiplier. If $\omega(g, g')$, as a function of the first variable can be viewed as defined over some coset space we can take f to be also defined over this coset space. This is the case for our ω_0 ; as a function of first variable it is defined over the coset space parametrized by boosts. If we call $v = -\frac{P}{m}$ we recover the standard representation.

The whole procedure can be applied to the quantum case; ω has the same property as ω_0 – it depends on the first variable only through boost v . In this way we arrive at the following form of (unitary) representation for twodimensional quantum Galilei group (24) :

$$\varrho : f(p) \rightarrow e^{-i\kappa \ln\left(1+\frac{p^2}{2m\kappa}\right)} \otimes e^{i\frac{p^2}{1+\frac{p^2}{2m\kappa}} \otimes a} f(p \otimes I - I \otimes mv). \quad (32)$$

Let us note that this representation is well defined only for $\kappa > 0$; the same phenomenon occurs for deformed Poincare group [9].

The above procedure can be extended to fourdimensional case. Moreover, using duality relations group \leftrightarrow algebra, one can find the infinitesimal form of representation and, consequently, the algebra obeyed by generators. This should indicate the way of constructing the *quantum-mechanical* extension of quantum Galilei group – in analogy with classical case. Another problem is how to obtain the representation of quantum Galilei group from those of deformed Poincaré group. All these questions will be addressed to in subsequent publication.

APPENDIX

We shall show that the commutation rules (11) can be obtained by quantizing the Poisson structure implied by commutator σ .

The classical duality group \leftrightarrow algebra is defined as

$$\langle \Phi, X \rangle = -i \frac{d}{dt} \Phi(e^{itX})|_{t=0} \quad (A1)$$

which implies equations (13).

We fix the following order of factors in monomials belonging to the universal covering of Galilei algebra : $\vec{M}, \vec{L}, \vec{P}, P_0$. The standard duality rules

$$\langle \Phi, XY \rangle = \langle \Delta \Phi, X \otimes Y \rangle, \quad \langle \Phi \Psi, X \rangle = \langle \Phi \otimes \Psi, \Delta X \rangle$$

give, in particular

$$\begin{aligned} \langle R_{ij}, J_{n_1} \dots J_{n_k} X \rangle &= (-i)^k \delta_{XI} \varepsilon_{il_1 n_1} \varepsilon_{l_1 l_2 n_2} \dots \varepsilon_{l_{k-1} j n_k}, \\ \langle \tau, X H^k \rangle &= i \delta_{XI} \delta_{k,1}, \\ \langle v_i, J_{n_1} \dots J_{n_k} L_m \rangle &= -i \langle R_{im}, J_{n_1} \dots J_{n_k} \rangle, \\ \langle a_i, J_{n_1} \dots J_{n_k} P_m \rangle &= -i \langle R_{im}, J_{n_1} \dots J_{n_k} \rangle, \\ \langle a_i, J_{n_1} \dots J_{n_k} L_m P_0 \rangle &= \langle R_{im}, J_{n_1} \dots J_{n_k} \rangle. \end{aligned} \quad (A2)$$

As an example let us calculate the Poisson bracket $\{R_n^m, a^r\}$:

$$\langle \{R_n^m, a^r\}, X \rangle = (-i) \langle R_n^m \otimes a^r, \sigma(X) \rangle. \quad (A3)$$

Using duality rules and the definition of cocommutator we readily infer that the right-hand side of (A3) does not vanish only for $X = M^k L_i$, where M^k stands for the product of k M 's with arbitrary indices. A straightforward calculation based on equations (A1), (A2), (9) and the duality rules gives

$$\langle \{R_n^m, a^r\}, M^k L_i \rangle = \frac{1}{\kappa} \langle \delta_{mr} v^p R_n^p - v^m R_n^r, M^k L_i \rangle. \quad (A4)$$

Now, we easily check that

$$\langle \delta_{mr} v^p R_n^p - v^m R_n^r, X \rangle = 0$$

for $X \neq M^k L_i$; therefore

$$\{R_n^m, a^r\} = -\frac{1}{\varkappa} (v^m R_n^r - \delta_{mr} v^p R_n^p) .$$

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